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Introduction

In this talk I introduce the basilar notions of modular forms. Starting with
the concept of congruence subgroup, I give the definition of modular form
and fundamental domain for such subgroups. Then I focus on the algebras of
modular forms and how to decompose them in some particular cases. Finally,
I give another decomposition of the cusp forms in the so-called old- and
new-forms through a description of the Hecke operators. If it remains time,
I will also give some examples.
1 Congruence subgroups

First of all, we define a class of subgroups that play a big role in the theory.

**Definition 1.1.** Fix an integer $N \geq 1$. The principal congruence subgroup of level $N$ is the group

$$\Gamma(N) := \{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \}.$$ 

A subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$ is called a congruence subgroup if there is an integer $N \geq 1$ such that $\Gamma(N) \subseteq \Gamma$.

**Remark 1.2.** For $N \geq 1$, the reduction map $\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective and $\Gamma(N) = \ker(\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z}))$. In particular, $\Gamma(N) \leq \text{SL}_2(\mathbb{Z})$.

Now, we show some examples.

**Example 1.3.**

$$\Gamma_0(N) := \{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \}$$

called the Hecke congruence subgroup of level $N$.

$$\Gamma_1(N) := \{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \}$$

And so, we have: $\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N)$.

In particular, for $N = 1$, we have $\Gamma(1) = \Gamma_1(1) = \Gamma_0(1) = \text{SL}_2(\mathbb{Z})$.

Notice that we have a group isomorphism $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ defined by $(a/b) \mapsto d \mod N$ with kernel $\Gamma_1(N)$. In other words, we have $\Gamma_1(N) \leq \Gamma_0(N)$ and $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$.

**Remark 1.4.**

1. $-I := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma_0(N)$ for any $N \geq 1$, but $-I \notin \Gamma_1(N)$ for any $N \geq 3$

2. if $N \mid M$, then $\Gamma(M) \subseteq \Gamma(N)$.

3. if $\Gamma \subset \text{SL}_2(\mathbb{Z})$ is a congruence subgroup, then $[\text{SL}_2(\mathbb{Z}) : \Gamma] < \infty$.

2 Modular forms

Let consider $\mathcal{H} := \{ z \in \mathbb{C} \mid \text{Im} z > 0 \}$ called the complex upper half plane endowed with the action of $\text{GL}_2^+(\mathbb{R}) := \{ \gamma \in \text{GL}_2(\mathbb{R}) \mid \det \gamma > 0 \}$ as follows: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define

$$\gamma.z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.z := \frac{az + b}{cz + d}.$$
First, note that if \( z \in \mathcal{H} \), then \( \text{Im}(\gamma.z) = \frac{\det \gamma}{|cz+d|^2} \cdot \text{Im} z > 0 \), so it defines a (well-defined) left action called \textit{fractional linear transformation} (or \textit{M"obius transformation}).

**Remark 2.1.**
1. \(-I\) acts trivially on \( \mathcal{H} \), so one can consider an action of \( \text{PSL}_2(\mathbb{Z}) := \text{SL}_2(\mathbb{Z})/\{\pm I\} \).
2. The action of \( \text{SL}_2(\mathbb{R}) \) on \( \mathcal{H} \) is transitive: if \( x+iy \in \mathcal{H} \) (i.e., \( y > 0 \)), then \( \left( \frac{\sqrt{y} x/\sqrt{y}}{0 1/\sqrt{y}} \right) i = x+iy \) (this remark is important for the generalizations like automorphic forms).

Let us consider \( \mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\} \). It is endowed with a (left) action of \( \text{SL}_2(\mathbb{Z}) \) as follows: if \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \) and \( x \in \mathbb{P}^1(\mathbb{Q}) \), define \( \gamma(x) = \frac{ax+b}{cx+d} \) with the ‘usual’ convention that \( \gamma.x = \infty \) if \( cx+d = 0 \), and

\[
\gamma.\infty = \begin{cases} 
  a/c & \text{if } c \neq 0 \\
  \infty & \text{if } c = 0 
\end{cases}
\]

**Remark 2.2.** \( \text{SL}_2(\mathbb{Z}) \) acts transitively on \( \mathbb{P}^1(\mathbb{Q}) \).

**Definition 2.3.** Let \( \Gamma \subseteq \text{SL}_2(\mathbb{Z}) \) be a subgroup. A \textit{cusp} of \( \Gamma \) is an equivalence class of elements of \( \mathbb{P}^1(\mathbb{Q}) \) under the action of \( \Gamma \), and the set of cusps is denoted by \( \Gamma \setminus \mathcal{H} \).

**Example 2.4.**
1. \( \text{SL}_2(\mathbb{Z}) \) has a unique cusp (\( \infty \)) since it acts transitively on \( \mathbb{P}^1(\mathbb{Q}) \). In particular, the stabilizer (or \textit{isotropy subgroup}) of \( \infty \) in \( \text{SL}_2(\mathbb{Z}) \) is the translations

\[
\text{SL}_2(\mathbb{Z})_{\infty} = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\}.
\]

2. If \( [\text{SL}_2(\mathbb{Z}) : \Gamma] < \infty \) (e.g. congruence subgroups), then \( \Gamma \) has finitely many cusps.

3. For any odd prime integer \( p \), \( \Gamma_0(p) \) has exactly two cusps (0 and \( \infty \)), while both \( \Gamma_1(p) \) and \( \Gamma(p) \) have \( (p-1)/2 \) cusps.

4. Denoting by \( P = \langle \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle = \text{SL}_2(\mathbb{Z})_{\infty} \) the parabolic subgroup of \( \text{SL}_2(\mathbb{Z}) \), then for any congruence subgroup \( \Gamma \) the map

\[
\Gamma \setminus \text{SL}_2(\mathbb{Z})/P \to \{ \text{cusps of } \Gamma \} \\
\Gamma \alpha P \mapsto \Gamma \alpha(\infty)
\]

is a bijection. Specifically, the map is \( \Gamma \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) P \mapsto \Gamma(a/c) \).
Let \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2^+(\mathbb{Q}) \) and \( k \geq 0 \) an integer. Take a complex valued function \( f : \mathcal{H} \to \mathbb{C} \) (for now just a map without request on continuity, ...).

Define the \( k \)-slash operator to be:

\[
(f|_k \gamma)(z) := (\det \gamma)^{k/2}(cz + d)^{-k}f(\gamma(z)).
\]

It is easy to see that this defines a right action on the complex valued map \( f \).

**Definition 2.5.** Let \( \Gamma \subseteq \text{SL}_2(\mathbb{Z}) \) be a congruence subgroup \( (\Gamma(N) \subseteq \Gamma \) for some \( N \)) and \( k \geq 0 \) be an integer.

A modular form of weight \( k \) for \( \Gamma \) is a function \( f : \mathcal{H} \to \mathbb{C} \) satisfying the following properties:

i) \( f \) is holomorphic on \( \mathcal{H} \);

ii) \( f|_k \gamma = f \) for any \( \gamma \in \Gamma \) (i.e., \( f(\gamma(z)) = (cz + d)^kf(z) \) for any \( z \in \mathcal{H} \));

iii) for any \( \sigma \in \text{SL}_2(\mathbb{Z}) \), we have: \( (f|_k \sigma)(z) = \sum_{n \geq 0} a_n q^n \), where \( a_n \in \mathbb{C} \) and \( q_N = q_N(z) := e^{2\pi i z/N} \) (holomorphicity of \( f \) at the cusps).

The \( \mathbb{C} \)-vector space of all modular forms of weight \( k \) for \( \Gamma \) is denoted by \( M_k(\Gamma) \).

We explain better the third condition.

Suppose \( \Gamma(N) \subseteq \Gamma \), then \( \gamma_N := \left( \begin{array}{cc} 1 & N \\ 0 & 1 \end{array} \right) \in \Gamma \); for the property ii) we have that \( f(z) = f(z + N) \) for any \( z \in \mathcal{H} \), and by holomorphicity on \( \mathcal{H} \) we get the Laurent expansion \( f(z) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i zn/N} \). We say that \( f \) is holomorphic at \( \infty \) if \( a_n = 0 \) for all \( n < 0 \).

For the other cusps, pick \( \sigma \in \text{SL}_2(\mathbb{Z}) \) and consider \( f|_k \sigma \); we get a similar expansion as before. Indeed, \( f|_k \sigma \) is invariant under \( \sigma^{-1} \Gamma \sigma \supseteq \sigma^{-1} \Gamma(N) \sigma = \Gamma(N) \) by normality, so \( f|_k \sigma \) is invariant under the matrix \( \gamma_N \), and then expand its Laurent series. So, we say that \( f \) is holomorphic at the cusp \( c \), for \( c = \sigma, \infty \), if \( f|_k \sigma \) is holomorphic at \( \infty \).

Now, it remains to show that if \( \sigma_1, \sigma_2 \in \text{SL}_2(\mathbb{Z}) \) satisfy that \( \sigma_1 \infty = \sigma_2 \infty \) and \( \sigma_2 \infty \) are \( \Gamma \)-equivalent (i.e., there is \( \gamma \in \Gamma \) such that \( \gamma \sigma_1 \infty = \sigma_2 \infty \)), then:

\[
\text{if } f|_k \sigma_1 \text{ satisfies iii) } \iff \text{if } f|_k \sigma_2 \text{ satisfies iii).}
\]

In particular, we are imposing only finitely many conditions at the cusps.

Finally, if \( f(z + h) = f(z) \) for some \( h \geq 1 \) and \( h < N \), then by group theory we have that \( h|N \) and via a direct computation we get that the Fourier expansion involves only powers of \( q_h = e^{2\pi i z/h} \). The smallest \( h \) that \( \Gamma(h) \subseteq \Gamma \) is called the level of the modular form \( f \). In particular, if \( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \in \Gamma \) (e.g. if \( \Gamma = \Gamma_1(N) \) or \( \Gamma_0(N) \)), then \( f(z + 1) = f(z) \) for any \( z \in \mathcal{H} \), so we can write \( f(z) = \sum_{n \geq 0} a_n q^n \) with \( q = q_1 := e^{2\pi i z} \), called the \( q \)-expansion of \( f \) and the \( a_n \)'s are called the Fourier coefficients of \( f \).
Example 2.6. 1. Put $T := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Since $\text{SL}_2(\mathbb{Z}) =< \pm T, \pm S \mid S^2 = (ST)^3 = 1 >$, we have that $f$ satisfies condition ii) for $\text{SL}_2(\mathbb{Z})$ if and only if $f(z + 1) = f(z)$ and $f(-1/z) = z^k f(z)$.

2. If $-I \in \Gamma$ and $f \in M_k(\Gamma)$, then $f(z) = (-1)^k f(z)$, so in this case for $k$ odd we have $M_k(\Gamma) = \{0\}$.

Definition 2.7. A modular form $f \in M_k(\Gamma)$ is a cusp form if $a_\sigma = 0$ for any $\sigma \in \text{SL}_2(\mathbb{Z})$ (vanishing at the cusps).

The $\mathbb{C}$-vector subspace of all cusp forms of weight $k$ for $\Gamma$ is denoted by $S_k(\Gamma)$.

Remark 2.8. If $\Gamma' \subseteq \Gamma$, then $M_k(\Gamma) \subseteq M_k(\Gamma')$.

3 Examples

Now, we will see some examples of modular forms, especially when the level $N$ is 1.

3.1 Eisenstein series

Fix $\Gamma = \text{SL}_2(\mathbb{Z})$ and an even integer $k \geq 4$. Define the Eisenstein series of weight $k$ to be:

$$G_k(z) := \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(mz+n)^k}, \quad z \in \mathcal{H}$$

The sum is absolutely convergent and uniformly convergent on compact subset of $\mathcal{H}$, in particular we have $G_k \in M_k(\text{SL}_2(\mathbb{Z}))$.

Let $\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}$ be the Riemann zeta function for $\text{Re}(s) > 1$ and recall that $\zeta(s) \neq 0$ in this region. Then, one can show that:

$$G_k(z) = 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

where $\sigma_{k-1}(n) := \sum_{m \mid n, m \geq 1} m^{k-1}$.

This means that $G_k(z) \in M_k(\text{SL}_2(\mathbb{Z})) S_k(\text{SL}_2(\mathbb{Z}))$.

In particular, we can define the normalised Eisenstein series $E_k(z) := \frac{G_k(z)}{2\zeta(k)} = 1 + \ldots$.

For the other levels, one can similarly define Eisenstein series, but the story is more complicated...
3.2 The discriminant form

Remark 3.1. If \( f \in M_k(\Gamma) \) and \( g \in M_{k'}(\Gamma) \), then \( f \cdot g \in M_{k+k'}(\Gamma) \).

Set \( \Gamma = \text{SL}_2(\mathbb{Z}) \), \( g_2(z) := 60G_4(z) \), \( g_3(z) := 140G_6(z) \), then define the discriminant function to be \( \Delta : \mathcal{H} \to \mathbb{C}, \Delta(z) = g_2(z)^3 - 27g_3(z)^2 \).

It is a modular form of weight 12 and \( a_0(\Delta) = 0 \), therefore \( \Delta \in S_{12}(\text{SL}_2(\mathbb{Z})) \). In fact, \( \Delta(z) \neq 0 \) for all \( z \in \mathcal{H} \) (by elliptic curve’s theory).

4 Fundamental domain

For later use, we introduce the concept of fundamental domain and try to compute their volume.

Definition 4.1. Let \( \Gamma \subseteq \text{SL}_2(\mathbb{Z}) \) be a congruence subgroup. The modular curve for \( \Gamma \) is the quotient space \( Y(\Gamma) := \Gamma \backslash \mathcal{H} \).

In the next lecture, we will see that \( Y(\Gamma) \) is a Riemann surface which can be compactified to obtain \( X(\Gamma) = \Gamma \backslash \mathcal{H}^* \), with \( \mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) \). Now, we want to “see” the modular curve as a region in \( \mathcal{H} \).

Definition 4.2. Let \( \Gamma \subseteq \text{SL}_2(\mathbb{Z}) \) be a subgroup of finite index. A subset \( F \subset \mathcal{H} \) is called a fundamental domain for \( \Gamma \) if:

- \( F \) is open;
- for any \( z \in \mathcal{H} \) there is \( \gamma \in \Gamma \) such that \( \gamma.z \in \overline{F} \);
- if \( z, z' \in F \) are \( \Gamma \)-equivalent (there is \( \gamma \in \Gamma \) for which \( z' = \gamma.z \)), then \( z = z' \) and \( \gamma = \pm I \).

Proposition 4.3. The full modular group \( \text{SL}_2(\mathbb{Z}) \) admits the fundamental domain

\[ F := \{ z \in \mathcal{H} \mid |\text{Re } z| < 1/2, |z| > 1 \} \]

Any congruence subgroup \( \Gamma \) admits a fundamental domain. In particular, if \( \{ \pm \} \Gamma \backslash \text{SL}_2(\mathbb{Z}) = \{ \gamma_1, \ldots, \gamma_r \} \) are representatives, then \( F(\Gamma) = \bigcup_{i=1}^r \gamma_i F \).

Define the hyperbolic measure on the upper half plane,

\[ d\mu(z) = \frac{dx dy}{y^2}, \quad z = x + iy \in \mathcal{H}. \]

This is invariant under the automorphism group \( \text{GL}_2^+(\mathbb{R}) \) of \( \mathcal{H} \), meaning \( d\mu(\alpha(z)) = d\mu(z) \) for any \( \alpha \in \text{GL}_2^+(\mathbb{R}) \) and \( z \in \mathcal{H} \), and in particular \( d\mu \) is \( \text{SL}_2(\mathbb{Z}) \)-invariant. Since the set \( \mathbb{P}^1(\mathbb{Q}) \) is countable it has measure zero, and
so $d\mu$ suffices for integrating over the extended upper half plane $\mathcal{H}^*$. Notice that the volume of $X(1) := X(SL_2(\mathbb{Z}))$ is finite:

$$V_{SL_2(\mathbb{Z})} = \int_{X(1)} \frac{dx dy}{y^2} = \int_{-1/2}^{1/2} dx \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2}$$

$$= \int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}} = \arcsin \left( \frac{1/2}{1/2} \right) = \frac{\pi}{3}$$

In particular, for any congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$, its volume is finite and is given by (see §8):

$$V_\Gamma = |SL_2(\mathbb{Z}) : \{ \pm I \} \Gamma| V_{SL_2(\mathbb{Z})}$$

5 The algebra of modular forms

Inspired by the remark 3.1, consider the direct sum $M(\Gamma) := \bigoplus_{k \geq 0} M_k(\Gamma)$ which is a graded ring. In particular, by Liouville’s theorem we have $M_0(\Gamma) = \mathbb{C}$ and so $S_0(\Gamma) = 0$; therefore, the algebra of modular forms $M(\Gamma)$ is a graded $\mathbb{C}$-algebra.

Similarly, we can define $S(\Gamma) := \bigoplus_{k \geq 0} S_k(\Gamma)$, which is a graded ideal of $M(\Gamma)$. Indeed, if $\{\alpha_i\}_{i=1}^r$ is a set of representatives of $\Gamma \backslash P^1(\mathbb{Q})$, then just consider the kernel of the $\mathbb{C}$-algebra morphism $M(SL_2(\mathbb{Z})) \rightarrow \mathbb{C}^r$ sending $f \mapsto (a_0(f|_{\alpha_i}))_i$.

For $\Gamma = SL_2(\mathbb{Z})$, it is well-known that $E_4$ and $E_6$ generate $M(SL_2(\mathbb{Z}))$; moreover, $E_4$ and $E_6$ are algebraically independent. In other word, the $\mathbb{C}$-vector space $M_k(SL_2(\mathbb{Z}))$ has a basis consisting of the monomials $E_4^\alpha E_6^\beta$ with $4\alpha + 6\beta = k$.

5.1 Dimension formulas

The $\mathbb{C}$-vector space of modular forms of a given weight for a congruence subgroup $\Gamma$ is always finite-dimensional. In general, we can estimate its dimension by the following formula (see [BGHZ08, pg. 12, Prop. 3]):

$$\dim \mathbb{C} M_k(\Gamma) \leq \frac{k \cdot V_\Gamma}{4\pi} + 1$$

for any $k \in \mathbb{Z}$. It easily follows that $\dim \mathbb{C} M_k(\Gamma) = 0$ for any $k < 0$ and $M_0(\Gamma) = \mathbb{C}$. In particular, for the case of level 1 there exists a precise computation:

$$\dim \mathbb{C} M_k(SL_2(\mathbb{Z})) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \equiv 2 \mod 12 \\ \left\lfloor \frac{k}{12} \right\rfloor + 1 & \text{if } k \not\equiv 2 \mod 12 \end{cases}$$

In the next lesson, we will see how to relate a geometrical invariant of the modular curve $X(\Gamma)$, the genus, to the exact dimension of modular forms $M_k(\Gamma)$.
6 Decomposing $M_k(\Gamma_1(N))$

Let $\Gamma' \subseteq \Gamma$ be congruence subgroups, $f \in M_k(\Gamma')$, $\gamma \in \Gamma$, then $f|_{k\gamma}$ depends only on the coset $\Gamma'\gamma$ (for $\gamma' \in \Gamma'$, $f|_{k\gamma'} = f|_{k\gamma}$).

Now, assume $\Gamma' \trianglelefteq \Gamma$. We get a group homomorphism (representation of $\Gamma/\Gamma'$)

$$\rho: \Gamma/\Gamma' \to \text{Aut}_\mathbb{C}(M_k(\Gamma'))$$

$$\Gamma'\gamma \mapsto (f \mapsto f|_{k\gamma}).$$

If $\chi: \Gamma/\Gamma' \to \mathbb{C}^\times$ be a character and define the $\mathbb{C}$-vector subspace

$$M_k(\Gamma, \chi) := \{ f \in M_k(\Gamma) \mid \rho(\gamma)(f) = \chi(\gamma)f \ \forall \gamma \in \Gamma \}.$$

In particular, if $\Gamma/\Gamma'$ is abelian, by representation theory we have that

$$M_k(\Gamma') = \bigoplus_{\chi: \Gamma/\Gamma' \to \mathbb{C}} M_k(\Gamma, \chi).$$

Example 6.1. Put $\Gamma = \Gamma_0(N)$ and $\Gamma' = \Gamma_1(N)$ and recall that $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$. Therefore, a character of $\Gamma_0(N)/\Gamma_1(N)$ identifies with a Dirichlet character $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. For a Dirichlet character $\chi$ modulo $N$, define the $\chi$-eigenspace of $M_k(\Gamma_1(N))$,

$$M_k(N, \chi) := M_k(\Gamma_0(N), \chi) = \{ f \in M_k(\Gamma_1(N)) \mid f|_{k\gamma} = \chi(\gamma)f \ \forall \gamma \in \Gamma \},$$

we have that

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times} M_k(N, \chi)$$

where $f_{\chi} := \frac{1}{\varphi(N)} \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(d)f|_{k\gamma_d}$, with $\varphi$ the totient Euler function and $\gamma_d \in \Gamma_0(N)$ such that $\gamma_d \equiv \begin{pmatrix} * & * \\ * & d \end{pmatrix} \mod N$.

Remark 6.2. • $M_k(N, \chi_{\text{triv}}) = M_k(\Gamma_0(N))$, where $\chi_{\text{triv}}: \bar{d} \mapsto 1$;

• $M_k(N, \chi) = 0$ if $\chi(-1) \neq (-1)^k$.

7 Hecke operators

In this section, we will see how to relate modular forms with different level, and to develop a theory in order to find a canonical basis for the space of cusp forms $S_k(\Gamma_1(N))$. 


7.1 The double coset operator

Let $\Gamma_1$ and $\Gamma_2$ be two congruence subgroups of $\text{SL}_2(\mathbb{Z})$. Then $\Gamma_1$ and $\Gamma_2$ are subgroups of $\text{GL}^+_2(\mathbb{Q})$, the group of 2-by-2 matrices with rational entries and positive determinant. For each $\alpha \in \text{GL}^+_2(\mathbb{Q})$ the set

$$\Gamma_1 \alpha \Gamma_2 = \{ \gamma_1 \alpha \gamma_2 \mid \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2 \}$$

is a double coset in $\text{GL}^+_2(\mathbb{Q})$.

The group $\Gamma_1$ acts on the double coset $\Gamma_1 \alpha \Gamma_2$ by left multiplication, partitioning it into orbits. A typical orbit is $\Gamma_1 \beta$ with representative $\beta = \gamma_1 \alpha \gamma_2$, and the orbit space $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ is thus a disjoint union $\bigcup \Gamma_1 \beta_j$ for some choice of representatives $\beta_j$. It is easy to see ([DS05, Lemma 5.1.2]) that the orbit space $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ is finite. So, we can define the action of the double coset on modular forms.

For $\beta \in \text{GL}^+_2(\mathbb{Q})$ and $k \in \mathbb{Z}$, the weight-$k$ operator on functions $f : \mathcal{H} \to \mathbb{C}$ is given by

$$(f[\beta]_k)(z) = (\det \beta)^{k-1}(cz + d)^{-k} f(\beta(z)), \quad z \in \mathcal{H}.$$

**Remark 7.1.** Notice that for elements $\beta \in \text{SL}_2(\mathbb{Z})$ we have that the $k$-slash operator is exactly equal to the weight-$k$ operator,

$$f | k \beta = f[\beta]_k \quad \text{for all } \beta \in \text{SL}_2(\mathbb{Z}).$$

In general, for elements $\beta \in \text{GL}^+_2(\mathbb{Q})$, we have the following relation between the two operators:

$$f[\beta]_k = (\det \beta)^{k/2-1} f | k \beta.$$

**Definition 7.2.** For congruence subgroups $\Gamma_1$ and $\Gamma_2$ of $\text{SL}_2(\mathbb{Z})$ and $\alpha \in \text{GL}^+_2(\mathbb{Q})$, the weight-$k$ $\Gamma_1 \alpha \Gamma_2$ operator takes functions $f \in M_k(\Gamma_1)$ to

$$f[\Gamma_1 \alpha \Gamma_2]_k = \sum_j f[\beta_j]_k$$

where $\{\beta_j\}$ are orbit representatives for $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$, i.e. $\Gamma_1 \alpha \Gamma_2 = \bigcup_j \Gamma_1 \beta_j$ is a disjoint union.

**Remark 7.3.** The double coset operator is well defined, i.e., it is independent of how the $\beta_j$ are chosen.

Note that any $\gamma_2 \in \Gamma_2$ permutes the orbit space $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ given by $\Gamma_1 \beta \mapsto \Gamma_1 \gamma_2 \beta$ is well defined and bijective. So, if $\{\beta_j\}$ is a set of orbit representatives for $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ then $\{\beta_j \gamma_2\}$ is a set of orbit representatives as well. Thus,

$$(f[\Gamma_1 \alpha \Gamma_2]_k)[\gamma_2]_k = \sum_j f[\beta_j \gamma_2]_k = f[\Gamma_1 \alpha \Gamma_2]_k,$$
in other words $f[\Gamma_1 \alpha \Gamma_2]_k$ is weight-$k$ invariant under $\Gamma_2$. Moreover, $f[\Gamma_1 \alpha \Gamma_2]_k$ is holomorphic at the cusps, then the weight-$k$ $\Gamma_1 \alpha \Gamma_2$ operator takes a modular forms with respect to $\Gamma_1$ to modular forms respect to $\Gamma_2$. In particular, the double coset operator takes cusp forms to cusp forms.

Special cases of the double coset operator $[\Gamma_1 \alpha \Gamma_2]_k$ arise when:

1. $\Gamma_1 \supset \Gamma_2$. Taking $\alpha = I$ makes the double coset operator be $f[\Gamma_1 \alpha \Gamma_2]_k = f$, the natural inclusion of the subspace $M_k(\Gamma_1)$ in $M_k(\Gamma_2)$, an injection.

2. $\alpha^{-1} \Gamma_1 \alpha = \Gamma_2$. Here the double coset operator is $f[\Gamma_1 \alpha \Gamma_2]_k = f[\alpha]_k$, the natural translation from $M_k(\Gamma_1)$ to $M_k(\Gamma_2)$, an isomorphism.

3. $\Gamma_1 \subset \Gamma_2$. Taking $\alpha = I$ and letting $\{\gamma_{2,j}\}$ be a set of coset representatives for $\Gamma_1 \backslash \Gamma_2$ makes the double coset operator be $f[\Gamma_1 \alpha \Gamma_2]_k = \sum_j f[\gamma_{2,j}]_k$, the natural trace map that projects $M_k(\Gamma_1)$ onto its subspace $M_k(\Gamma_2)$ by symmetrizing over the quotient, a surjection.

In general, one can show that any double coset operator is a composition of these.

Finally, there is also a geometric interpretation of the double coset operator, but this is another story...

7.2 The diamond and the Hecke operators

To define the first type of Hecke operator, take any $\alpha \in \Gamma_0(N)$, set $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$, and consider the weight-$k$ double coset operator $[\Gamma_1 \alpha \Gamma_2]_k$. Since $\Gamma_1(N) \triangleleft \Gamma_0(N)$ this operator is case (2), translating each function $f \in M_k(\Gamma_1(N))$ to

$$f[\Gamma_1 \alpha \Gamma_2]_k = f[\alpha]_k, \quad \alpha \in \Gamma_0(N),$$

again in $M_k(\Gamma_1(N))$. Thus, the group $\Gamma_0(N)$ acts on $M_k(\Gamma_1(N))$, and since its subgroup $\Gamma_1(N)$ acts trivially, this is really an action of the quotient $(\mathbb{Z}/N\mathbb{Z})^\times$. The action of $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, determined by $d \mod N$ and denoted by $\langle d \rangle$, is

$$\langle d \rangle : M_k(\Gamma_1(N)) \rightarrow M_k(\Gamma_1(N))$$

given by

$$\langle d \rangle f = f[\alpha]_k \quad \text{for any } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ with } \delta \equiv d \mod N$$

This is the first type of Hecke operator, also called diamond operator. For any character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}$, the space $M_k(N, \chi)$ is precisely the $\chi$-eigenspace of the diamond operators:

$$M_k(N, \chi) = \{ f \in M_k(\Gamma_1(N)) \mid \langle d \rangle f = \chi(d)f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^\times \}.$$
That is, the diamond operator \( \langle d \rangle \) respects the decomposition \( M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(N, \chi) \), operating on the eigenspace associated to each character \( \chi \) as multiplication by \( \chi(d) \).

**Remark 7.4.** if \( f \in M_k(\Gamma_0(N)) \), then \( \langle d \rangle \) acts trivially on \( f \).

The second type of Hecke operator is also a weight-\( k \) double coset operator \([\Gamma_1 \alpha \Gamma_2]_k\) where again \( \Gamma_1 = \Gamma_2 = \Gamma_1(N) \), but now \( \alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \), \( p \) prime integer.

This operator is denoted \( T_p \). Thus

\[
T_p : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N)), \quad p \text{ prime}
\]

is given by

\[
T_p f = f[\Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_2]_k
\]

One can see that the double coset is

\[
\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) = \{ \gamma \in M_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & p \end{pmatrix} \mod N, \det \gamma = p \},
\]

so, in fact \( \alpha \) can be replaced by any matrix in this double coset in the definition of \( T_p \).

**Proposition 7.5.** Let \( N \in \mathbb{Z}_{>0} \), let \( \Gamma_1 = \Gamma_2 = \Gamma_1(N) \), and let \( \alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \) where \( p \) is prime. The operator \( T_p = [\Gamma_1 \alpha \Gamma_2]_k \) on \( M_k(\Gamma_1(N)) \) is given by

\[
T_p f = \begin{cases} 
\sum_{j=0}^{p-1} f[\begin{pmatrix} 1 + j & 0 \\ 0 & p \end{pmatrix}]_k & \text{if } p \mid N \\
\sum_{j=0}^{p-1} f[\begin{pmatrix} 1 + j & 0 \\ 0 & p \end{pmatrix}]_k + f[\begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}]_k & \text{if } p \not{\mid} N, \text{ where } mp - nN = 1.
\end{cases}
\]

**Remark 7.6.** Usually, in literature, when \( p \mid N \), the operator \( T_p \) is denoted by \( U_p \).

The Hecke operators commute.

**Proposition 7.7.** Let \( d \) and \( e \) be elements of \((\mathbb{Z}/N\mathbb{Z})^\times\), and let \( p \) and \( q \) be distinct primes. Then

(a) \( \langle d \rangle T_p = T_p \langle d \rangle \),

(b) \( \langle d \rangle \langle e \rangle = \langle e \rangle \langle d \rangle = \langle de \rangle \),

(c) \( T_p T_q = T_q T_p \).
Now, define the Hecke operators \( \langle n \rangle \) and \( T_n \) for any integer \( n > 0 \).

For \( n \in \mathbb{Z}_{>0} \) with \( (n, N) = 1 \), \( \langle n \rangle \) is determined by \( n \mod N \). For \( n \in \mathbb{Z}_{>0} \) with \( (n, N) > 1 \), define \( \langle n \rangle = 0 \), the zero operator on \( M_k(\Gamma_1(N)) \).

The mapping \( n \mapsto \langle n \rangle \) is totally multiplicative, i.e., \( \langle nm \rangle = \langle n \rangle \langle m \rangle \) for any \( n, m \in \mathbb{Z}_{>0} \).

To define \( T_n \), set \( T_1 = 1 \) the identity operator; \( T_p \) is already defined for primes \( p \). For prime powers, define inductively

\[
T_{p^r} = T_p T_{p^{r-1}} - p^{k-1}(p) T_{p^{r-2}}, \quad \text{for } r \geq 2,
\]

and note that inductively on \( r \) and \( s \) starting from the previous proposition, \( T_{p^r} T_{q^s} = T_{q^s} T_{p^r} \) for distinct primes \( p \) and \( q \). Extend the definition multiplicatively to \( T_n \) for all \( n \),

\[
T_n = \prod T_{p_i^{e_i}}, \quad \text{where } n = \prod p_i^{e_i},
\]

so that the \( T_n \) all commute by Proposition 7.7 and \( T_{nm} = T_n T_m \) if \( (n, m) = 1 \).

**Remark 7.8.** If \( p \mid N \), the above definition is easier: \( U_{p^r} = U_p U_{p^{r-1}} \) for \( r \geq 2 \).

The next result describes the effect of the operators \( T_n \) on Fourier coefficients.

**Proposition 7.9.** Let \( f \in M_k(\Gamma_1(n)) \) have Fourier expansion

\[
f(z) = \sum_{m \geq 0} a_m(f) q^m \quad \text{where } q = e^{2\pi i z}.
\]

Then for all \( n \in \mathbb{Z}_{>0} \), \( T_n f \) has Fourier expansion \( (T_n f)(z) = \sum_{m \geq 0} a_m(T_n f) q^m \) where

\[
a_m(T_n f) = \sum_{d \mid (m, n)} d^{k-1} a_{mn/d^2}(\langle d \rangle f).
\]

In particular, if \( f \in M_k(N, \chi) \) then

\[
a_m(T_n f) = \sum_{d \mid (m, n)} \chi(d) d^{k-1} a_{mn/d^2}(f). \quad (7.1)
\]

### 8 The Petersson inner product

Recall the hyperbolic measure on the upper half plane,

\[
d\mu(z) = \frac{dx dy}{y^2}, \quad z = x + iy \in \mathbb{H}.
\]
Given a continuous and bounded function \( \varphi : \mathcal{H} \to \mathbb{C} \) and any \( \alpha \in \text{SL}_2(\mathbb{Z}) \), the integral \( \int_X \varphi(\alpha(z))d\mu \) converges.

Let \( \Gamma \) be a congruence subgroup. In order to define the integral over its modular curve \( X(\Gamma) \), take \( \{\alpha_j\} \subset \text{SL}_2(\mathbb{Z}) \) representatives of the coset space \( \{ \pm I \} \Gamma \backslash \text{SL}_2(\mathbb{Z}) \), meaning that the union \( \text{SL}_2(\mathbb{Z}) = \bigcup_j \{ \pm I \} \Gamma \alpha_j \) is disjoint. If the function \( \varphi \) is also \( \Gamma \)-invariant, then the sum \( \sum_j \int_X \varphi(\alpha_j(z))d\mu \) is independent of the choice of coset representatives \( \alpha_j \). Since \( d\mu \) is \( \text{SL}_2(\mathbb{Z}) \)-invariant, the sum is \( \int_{\bigcup \alpha_j(\mathcal{F})} \varphi(z)d\mu \): this is the definition for \( \int_X(\Gamma) \).

**Definition 8.1.** Let \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \) be a congruence subgroup. The Petersson inner product, 
\[
\langle \cdot, \cdot \rangle_\Gamma : S_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C},
\]
is given by
\[
\langle f, g \rangle_\Gamma = \frac{1}{V_\Gamma} \int_{X(\Gamma)} f(z)\overline{g(z)}(\text{Im}(z))^k d\mu.
\]

For \( f, g \in S_k(\Gamma) \), set \( \varphi(z) := f(z)\overline{g(z)}(\text{Im}(z))^k \) with \( z \in \mathcal{H} \). It is clearly continuous. It is \( \Gamma \)-invariant because for \( \gamma \in \Gamma \):
\[
\varphi(\gamma, z) = f(\gamma, z)\overline{g(\gamma, z)}(\text{Im}(\gamma, z))^k
\]
\[
= (f(\gamma, z)) (cz + d)^k \overline{(g(\gamma, z)) (cz + d)^k} (\text{Im}(z))^k |cz + d|^{-2k}
\]
\[
= (f(\gamma, z)) (\overline{g(\gamma, z)}) (\text{Im}(z))^k
\]
\[
= f(z)\overline{g(z)}(\text{Im}(z))^k \quad \text{because } f \text{ and } g \text{ are weight-}k \text{ invariant under } \Gamma.
\]

In order to show that \( \varphi \) is bounded on \( \mathcal{H} \), it is enough to show it for the fundamental domain \( \bigcup \alpha_i(\mathcal{F}) \) of \( \Gamma \), but since this is a finite union, we need to show that \( \varphi \circ \alpha \) is bounded on \( \mathcal{F} \) for any \( \alpha \in \text{SL}_2(\mathbb{Z}) \). Being continuous, \( \varphi \circ \alpha \) is certainly bounded on any compact subset of \( \mathcal{F} \). As for neighborhoods \( \{\text{Im}(z) > y\} \) of \( \infty \), first note the Fourier expansions
\[
(f(\alpha)_k(z)) = \sum_{n \geq 1} a_n(f(\alpha)_k) q_h^n \quad (g(\alpha)_k(z)) = \sum_{n \geq 1} a_n(g(\alpha)_k) q_h^n,
\]
where \( q_h = e^{2\pi i z/h} \) for some \( h \in \mathbb{Z}_{>0} \). Each of these is of the order \( q_h \) (written \( O(q_h) \)) as \( \text{Im}(z) \to \infty \). Thus as above,
\[
\varphi(\alpha(z)) = (f(\alpha)_k(z)) (\overline{g(\alpha)_k}(z)) (\text{Im}(z))^k = O(q_h)^2(\text{Im}(z))^k
\]
by the Fourier expansion. Since \( |q_h| = e^{-2\pi \text{Im}(z)/h} \) and exponential decay dominates polynomial growth, \( \varphi \circ \alpha \to 0 \) as \( \text{Im}(z) \to \infty \) and \( \varphi \circ \alpha \) is bounded on \( \mathcal{F} \) as desired. This shows that the definition of Petersson inner product is well-defined and convergent.

Clearly, the product is linear in \( f \), semilinear in \( g \), Hermitian-symmetric, and positive definite. The normalized factor \( 1/V_\Gamma \) ensures that if \( \Gamma' \subset \Gamma \) then \( \langle \cdot, \cdot \rangle_{\Gamma'} = \langle \cdot, \cdot \rangle_{\Gamma} \) on \( S_k(\Gamma) \).
8.1 Adjoint of the Hecke Operators

Recall that if $V$ is an inner product space and $T$ is a linear operator on $V$, then the adjoint $T^*$ is the linear operator on $V$ defined by the condition

$$\langle Tv, w \rangle = \langle v, T^*w \rangle$$

for all $v, w \in V$.

Recall also that the operator $T$ is called normal when it commutes with its adjoint.

**Proposition 8.2.** Let $\Gamma \subset \text{SL}_2(\mathbb{Z})$ be a congruence subgroup, and let $\alpha \in \text{GL}_2^+(\mathbb{Q})$. Set $\alpha' = \det(\alpha)\alpha^{-1}$. Then

(a) If $\alpha^{-1}\Gamma\alpha \subset \text{SL}_2(\mathbb{Z})$ then for all $f \in S_k(\Gamma)$ and $g \in S_k(\alpha^{-1}\Gamma\alpha)$,

$$\langle f[\alpha]_k, g \rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g[\alpha']_k \rangle_{\Gamma}.$$

(b) For all $f, g \in S_k(\Gamma)$,

$$\langle f[\Gamma\alpha \Gamma]_k, g \rangle = \langle f, g[\Gamma\alpha' \Gamma]_k \rangle.$$

In particular, if $\alpha^{-1}\Gamma\alpha = \Gamma$ then $[\alpha]_k = [\alpha']_k$, and in any case $[\Gamma\alpha \Gamma]_k^* = [\Gamma\alpha' \Gamma]_k$.

**Theorem 8.3.** In the inner product space $S_k(\Gamma_1(N))$, the Hecke operators $\langle p \rangle$ and $T_p$ for $p \nmid N$ have adjoints:

$$\langle p \rangle^* = \langle p \rangle^{-1} \quad \text{and} \quad T_p^* = \langle p \rangle^{-1}T_p.$$

Thus, the Hecke operators $\langle n \rangle$ and $T_n$ for $n$ relatively prime to $N$ are normal.

From the Spectral Theorem of linear algebra, given a commuting family of normal operators on a finite-dimensional inner product space, the space has an orthogonal basis of simultaneous eigenvectors for the operators. Since each such vector is a modular form we say eigenform instead, and the result is

**Theorem 8.4.** The space $S_k(\Gamma_1(N))$ has an orthogonal basis of simultaneous eigenforms for the Hecke operators $\{\langle n \rangle, T_n \mid (n, N) = 1\}$.

The next sections will partly eliminate the restriction that $(n, N) = 1$. Note that when $(n, N) > 1$, we have $\langle n \rangle^* = 0^* = 0$.

9 Hecke eigenforms

We start this section on investigating the relations between forms of different levels.
9.1 Oldforms and Newforms

Recall that if $M \mid N$ then we have a natural inclusion $S_k(\Gamma_1(M)) \subset S_k(\Gamma_1(N))$.

Another way to embed $S_k(\Gamma_1(M))$ into $S_k(\Gamma_1(N))$ is by composing with the multiply-by-$d$ map, where $d$ is any factor of $N/M$. For any such $d$, let

$$\alpha_d := \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$$

so that $(f[\alpha_d]_k)(z) = d^{k-1} f(dz)$ for $f: \mathcal{H} \to \mathbb{C}$. The injective linear map $[\alpha_d]_k$ takes $S_k(\Gamma_1(M))$ to $S_k(\Gamma_1(N))$, lifting the level from $M$ to $N$.

Combining the observations so far, it is natural to distinguish the part of $S_k(\Gamma_1(N))$ coming from lower levels.

**Definition 9.1.** For each divisor $d$ of $N$, let $i_d$ be the map

$$i_d: S_k(\Gamma_1(N^{-1}d^2)) \rightarrow S_k(\Gamma_1(N))$$

$$(f, g) \mapsto f + g[\alpha_d]_k.$$

The subspace of oldforms at level $N$ is

$$S_k(\Gamma_1(N))^{\text{old}} = \sum_{p \mid N\text{prime}} i_p((S_k(\Gamma_1(N^{-1}p)))^2)$$

and the subspace of newforms at level $N$ is the orthogonal complement with respect to the Petersson inner product,

$$S_k(\Gamma_1(N))^{\text{new}} = (S_k(\Gamma_1(N))^{\text{old}})^\perp.$$ 

The Hecke operators respect the decomposition of $S_k(\Gamma_1(N))$ into old and new.

**Proposition 9.2.** The subspaces $S_k(\Gamma_1(N))^{\text{old}}$ and $S_k(\Gamma_1(N))^{\text{new}}$ are stable under the Hecke operators $T_n$ and $\langle n \rangle$ for all $n \in \mathbb{Z}_{>0}$.

**Corollary 9.3.** The spaces $S_k(\Gamma_1(N))^{\text{old}}$ and $S_k(\Gamma_1(N))^{\text{new}}$ have orthogonal bases of eigenforms for the Hecke operators away from the level, $\{T_n, \langle n \rangle \mid (n, N) = 1\}$.

9.2 Eigenforms

Let $M \mid N$ and $d \mid (N/M)$, $d > 1$. Thus $\Gamma_1(N) \subset \Gamma_1(M)$.

To normalize the scalar of the map $i_d$ to 1, define a variant

$$i_d = d^{1-k}[\alpha_d]_k: S_k(\Gamma_1(M)) \rightarrow S_k(\Gamma_1(N)),$$ 

$$(i_d f)(z) = f(dz),$$
acting on Fourier expansions as
\[ t_d : \sum_{n \geq 1} a_n q^n \mapsto \sum_{n \geq 1} a_n q^{dn}, \quad \text{where } q = e^{2\pi i z}. \]

This shows that if \( f \in S_k(\Gamma_1(N)) \) takes the form \( f = \sum_{p \mid N} t_pf_p \) with each \( f_p \in S_k(\Gamma_1(N/p)) \), and if the Fourier expansion of \( f \) is \( f(z) = \sum a_n(f) q^n \), then \( a_n(f) = 0 \) for all \( n \) such that \( (n, N) = 1 \). The main lemma in the theory of newforms is that the converse holds as well.

**Lemma 9.4** (Main Lemma). If \( f \in S_k(\Gamma_1(N)) \) has Fourier expansion \( f(z) = \sum a_n(f) q^n \) with \( a_n(f) = 0 \) whenever \( (n, N) = 1 \), then \( f \) takes the form \( f = \sum_{p \mid N} t_pf_p \) with each \( f_p \in S_k(\Gamma_1(N/p)) \).

To allow the discussion here to cover all the Hecke operators \( T_n \) and \( \langle n \rangle \) for all \( n \in \mathbb{Z}_{>0} \), we present another definition.

**Definition 9.5.** A nonzero modular form \( f \in M_k(\Gamma_1(N)) \) that is an eigenform for the Hecke operators \( T_n \) and \( \langle n \rangle \) for all \( n \in \mathbb{Z}_{>0} \) is a Hecke eigenform or simply an eigenform. The eigenform \( f(z) = \sum_{n \geq 0} a_n(f) q^n \) is normalized when \( a_1(f) = 1 \). A newform is a normalized eigenform in \( S_k(\Gamma_1(N))^{\text{new}} \).

**Remark 9.6.** In general, we have a decomposition \( M_k(\Gamma) = \mathcal{E}_k(\Gamma) \oplus S_k(\Gamma) \). The above definition wants to include also the Eisenstein series when they are nonzero.

We stress that, as in linear algebra, we don’t include the zero form as eigenform. We will prove that \( S_k(\Gamma_1(N))^{\text{new}} \) has an orthogonal basis of newforms.

Let \( f \in S_k(\Gamma_1(N)) \) be an eigenform for the Hecke operators \( T_n \) and \( \langle n \rangle \) with \( \langle n, N \rangle = 1 \). Thus, for all such \( n \) there exists eigenvalues \( c_n, d_n \in \mathbb{C} \) such that \( T_n f = c_n f \) and \( \langle n \rangle f = d_n f \). The map \( n \mapsto d_n \) defines a Dirichlet character \( \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \), and \( f \in S_k(N, \chi) \). Consequently, the formula (7.9) applies and says
\[ a_1(T_n f) = a_n(f) \quad \text{for all } n \in \mathbb{Z}_{>0}. \]  
(9.1)

Since \( f \) is an eigenform away from the level, also
\[ a_1(T_n f) = c_n a_1(f) \quad \text{when } \langle n, N \rangle = 1, \]
so
\[ a_n(f) = c_n a_1(f) \quad \text{when } \langle n, N \rangle = 1. \]

Thus, if \( a_1(f) = 0 \) then \( a_n(f) = 0 \) when \( \langle n, N \rangle = 1 \) and so \( f \in S_k(\Gamma_1(N))^{\text{old}} \) by the Main Lemma.

Now, assume that \( f \in S_k(\Gamma_1(N))^{\text{new}}, f \neq 0 \). Then \( f \not\in S_k(\Gamma_1(N))^{\text{old}} \) and the preceding paragraph shows that \( a_1(f) \neq 0 \), so we may assume \( f \) is normalized to \( a_1(f) = 1 \).
Theorem 9.7. Let $f \in S_k(\Gamma_1(N))^{\text{new}}$ be a nonzero eigenform for the Hecke operators $T_n$ and $\langle n \rangle$ for all $n$ with $(n, N) = 1$. Then:

(a) $f$ is a Hecke eigenform, i.e. an eigenform for $T_n$ and $\langle n \rangle$ for all $n \in \mathbb{Z}_{>0}$. A suitable scalar multiple of $f$ is a newform.

(b) If $\tilde{f}$ satisfies the same conditions as $f$ and has the same $T_n$-eigenvalues, then $\tilde{f} = cf$ for some constant $c$ (Multiplicity One property).

Proof. For any $m \in \mathbb{Z}_{>0}$ let $g_m := T_m f - a_m(f)f$, an element of $S_k(\Gamma_1(N))^{\text{new}}$ and an eigenform for the Hecke operators $T_n$ and $\langle n \rangle$ for $(n, N) = 1$. Compute that its first coefficient is

$$a_1(g_m) = a_1(T_m f) - a_1(a_m(f)f)$$

$$= a_m(f) - a_m(f) \quad \text{by (9.1) and because } a_1(f) = 1$$

$$= 0,$$

showing that $g_m \in S_k(\Gamma_1(N))^{\text{old}}$ by the argument of the preceding paragraph. So $g_m \in S_k(\Gamma_1(N))^{\text{new}} \cap S_k(\Gamma_1(N))^{\text{old}} = \{0\}$, i.e., $T_m f = a_m(f)f$. Since $m$ is arbitrary the discussion proves part (a).

All that remains to prove is that the set of newforms in the space $S_k(\Gamma_1(N))^{\text{new}}$ is linearly independent. To see this, suppose there is a nontrivial linear relation $\sum_{i=1}^n c_i f_i = 0$, $c_i \in \mathbb{C}$ with all $c_i$ nonzero and with as few terms as possible, necessarily at least two. For any prime $p$, applying $T_p - a_p(f_1)$ to the relation gives

$$\sum_{i=2}^n c_i(a_p(f_i) - a_p(f_1))f_i = 0.$$

This relation must be trivial since it has fewer terms, so $a_p(f_i) = a_p(f_1)$ for all $i$. Since $p$ is arbitrary this means that $f_i = f_1$ for all $i$, giving a contradiction since the original relation has at least two terms. \[\square\]

The newforms in each diamond operator eigenspace $S_k(\Gamma_1(N))^{\text{new}}$ therefore are an orthogonal basis of the eigenspace. The Multiplicity One property of newforms shows that the basis of $S_k(\Gamma_1(N))^{\text{new}}$ contains one element per eigenvalue where "eigenvalue" means a set of $T_n$-eigenvalues $\{c_n \mid n \in \mathbb{Z}_{>0}\}$. That is, each eigenspace for the $T_n$ operators is 1-dimensional. If $f \in S_k(\Gamma_1(N))$ is an eigenform then $f$ is old or new, never a hybrid $f = g + h$ with $g$ old and $h$ new and both nonzero.

Proposition 9.8 (Strong Multiplicity One). Let $g \in S_k(\Gamma_1(N))$ be a normalized eigenform. Then there is a newform $f \in S_k(\Gamma_1(M))^{\text{new}}$ for some $M \mid N$ such that $a_p(f) = a_p(g)$ for all $p \mid N$.  

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The level $M$ of the newform $f$ in the proposition is called the conductor of $f$. The result is analogous to the fact that any Dirichlet character $\chi \mod N$ has a corresponding primitive character $\chi_{\text{prim}} \mod M$ of $\chi$ such that $\chi_{\text{prim}}(p) = \chi(p)$ for all $p \nmid N$.

The last result of this section holds for all modular forms.

**Proposition 9.9.** Let $f \in M_k(N, \chi)$. Then $f$ is a normalized eigenform if and only if its Fourier coefficients satisfy the conditions:

1. $a_1(f) = 1$,
2. $a_{p^r}(f) = a_p(f)a_{p^{r-1}}(f) - \chi(p)p^{k-1}a_{p^{r-2}}(f)$ for all $p$ prime and $r \geq 2$,
3. $a_{mn}(f) = a_m(f)a_n(f)$ when $(m, n) = 1$.

The proposition does not say that any function $f(z) = \sum_n a_n(f)q^n$ with coefficients satisfying conditions (1), (2) and (3) is a normalized eigenform. The function need not to be a modular form at all.

**Proof.** The forward implication ($\Rightarrow$) follows from the definition of $T_n$. For the reverse implication ($\Leftarrow$), suppose $f$ satisfies the three conditions. Then $f$ is normalized, and to be an eigenform for all the Hecke operators it needs only satisfy $a_m(T_pf) = a_p(f)a_m(f)$ for all $p$ prime and $m \in \mathbb{Z}_{>0}$. If $p \nmid m$ then formula (9.1) gives $a_m(T_pf) = a_pm(f)$ and by third condition this is $a_p(f)a_m(f)$ as desired. On the other hand, if $p \mid m$ write $m = p^r m'$ with $r \geq 1$ and $p \nmid m'$. This time

$$a_m(T_pf) = a_{p^{r+1}m'}(f) + \chi(p)p^{k-1}a_{p^{r-1}m'}(f)$$

by formula (9.1)

$$= (a_{p^{r+1}}(f) + \chi(p)p^{k-1}a_{p^{r-1}}(f))a_{m'}(f)$$

by third condition

$$= a_p(f)a_{p^r}(f)a_{m'}(f)$$

by second condition

$$= a_p(f)a_m(f)$$

by third condition.

\[ \square \]

### 10 Algebraic eigenvalues

A extraordinary result is the following.

**Theorem 10.1.** Let $f \in S_2(\Gamma_1(N))$ be a normalized eigenform for the Hecke operators $T_p$. Then the eigenvalues $a_n(f)$ are algebraic integers.

To refine this result we need to view the Hecke operators as lying within an algebraic structure, not merely as a set.
Definition 10.2. The Hecke algebra over $\mathbb{Z}$ is the algebra of endomorphisms of $S_2(\Gamma_1(N))$ generated over $\mathbb{Z}$ by the Hecke operators,

\[ T_\mathbb{Z} := \mathbb{Z}[\{T_n, (n) \mid n \in \mathbb{Z}_{>0}\}] = \mathbb{Z}[T_p, (p), U_q \mid p, q \text{ primes, } p \nmid N, q \mid N]. \]

The Hecke algebra $T_\mathbb{C}$ over $\mathbb{C}$ is defined similarly.

Each level has its own Hecke algebra, but $N$ is omitted from the notation since it is usually written somewhere nearby. Clearly, any $f \in S_2(\Gamma_1(N))$ is an eigenform for all of $T_\mathbb{C}$ if and only if $f$ is an eigenform for all Hecke operators $T_p$ and $(d)$.

Remark 10.3. The Hecke algebra $T_\mathbb{C}$ is finite dimensional because so is the endomorphism ring of $S_2(\Gamma_1(N))$. In a similar way, one can see that $T_\mathbb{Z}$ is a finitely generated $\mathbb{Z}$-module.

Let $f(z) = \sum_{n \geq 1} a_n(f)q^n$ be a normalized eigenform, the eigenvalue homomorphism

\[ \lambda_f: T_\mathbb{Z} \to \mathbb{C}, \quad Tf = \lambda_f(T)f \]

therefore has as its image a finitely generated $\mathbb{Z}$-module. Since the image is $\mathbb{Z}[\{a_n(f) \mid n \in \mathbb{Z}_{>0}\}]$ this shows that even though there are infinitely many eigenvalues $a_n(f)$, the ring they generate has finite rank as a $\mathbb{Z}$-module. More specifically, letting

\[ I_f = \ker(\lambda_f) = \{T \in T_\mathbb{Z} \mid Tf = 0\} \]

gives a ring and $\mathbb{Z}$-module isomorphism

\[ T_\mathbb{Z}/I_f \cong \mathbb{Z}[\{a_n(f)\}]. \]

The image ring sits inside some finite-degree extension field of $\mathbb{Q}$, i.e. a number field. The rank of $T_\mathbb{Z}/I_f$ is the degree of this number field as an extension of $\mathbb{Q}$.

Definition 10.4. Let $f \in S_2(\Gamma_1(N))$ be a normalized eigenform, $f(z) = \sum_{n \geq 1} a_n(f)q^n$. The field $K_f = \mathbb{Q}(\{a_n\})$ generated by the Fourier coefficients of $f$ is called the number field of $f$.

Any embedding $\sigma: K_f \to \mathbb{C}$ conjugates $f$ by acting on its coefficients. That is, if $f(z) = \sum_{n \geq 1} a_n q^n$ then notating the action with a superscript,

\[ f^\sigma(z) = \sum_{n \geq 1} a_n^\sigma q^n. \]

In fact, this action produces another eigenform.
Theorem 10.5. Let $f$ be a weight-2 normalized eigenform of the Hecke operators, so that $f \in s_2(N, \chi)$ for some $N$ and $\chi$. Let $K_f$ be its number field. For any embedding $\sigma: K_f \hookrightarrow \mathbb{C}$ the conjugated $f^\sigma$ is also a normalized eigenform in $S_2(N, \chi)$ where $\chi^\sigma(n) = \chi(n)^\sigma$. If $f$ is a newform then so is $f^\sigma$.

Linearly combining the normalized eigenforms gives modular forms with coefficients in $\mathbb{Z}$.

Corollary 10.6. The space $S_2(\Gamma_1(N))$ has a basis of forms with rational integer coefficients.

11 Gauß sums

In this section we will give a brief recall of the propriety of the Dirichlet characters and of the Gauß sum attached to them.

Let $N \geq 1$ be an integer. A Dirichlet character modulo $N$ is a character $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ with the convention that $(\mathbb{Z}/1\mathbb{Z})^\times = \{0\}$. In the latter case, the only Dirichlet character modulo 1 is the trivial one $\chi_{\text{triv}}: n \mapsto 1$.

The dual group of $(\mathbb{Z}/N\mathbb{Z})^\times$ is the multiplicative group of Dirichlet characters modulo $N$, with pointwise multiplication and identity element the trivial character $\chi_{\text{triv}}$ modulo $N$. Since $(\mathbb{Z}/N\mathbb{Z})^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times$, we have that the number of Dirichlet characters is $\varphi(N)$, where $\varphi$ is the totient Euler function.

Moreover, thanks to the orthogonality of characters (or a direct computation) we have that

$$\sum_{n \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(n) = \begin{cases} \varphi(n) & \text{if } \chi = \chi_{\text{triv}} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{\chi \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(n) = \begin{cases} \varphi(n) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}.$$ 

Let $N \geq 1$ be an integer and $d$ be a positive divisor of $N$. Every Dirichlet character $\chi_d$ modulo $d$ lifts to a Dirichlet character $\chi_N$ modulo $N$ by reduction

$$(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times.$$ 

Thus, we define the conductor $f(\chi)$ of a Dirichlet character $\chi$ modulo $N$ to be the greatest common divisor of all divisors $d \mid N$ such that $\chi$ is induced by a character modulo $d$. Finally, a Dirichlet character $\chi$ modulo $N$ is called primitive if it is not induced by a Dirichlet character modulo $d$ for any proper divisors $d \mid N$, i.e. $f(\chi) = N$.

Every Dirichlet character modulo $N$ extends to a function $\chi: \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ where $\chi(n) = 0$ if $(n, N) \neq 1$, and then extends further to a function $\chi: \mathbb{Z} \to \mathbb{C}$ where $\chi(n) := \chi(n \mod N)$ for all $n \in \mathbb{Z}$. The last function is no
longer a homomorphism but it still is multiplicative: \( \chi(nm) = \chi(n)\chi(m) \) for all \( n, m \in \mathbb{Z} \).

Let \( \chi \) be a Dirichlet character modulo \( N \) and \( b \in \mathbb{Z} \). The Gauß sum attached to \( \chi \) is the complex number
\[
\tau(b, \chi) := \sum_{n=0}^{N-1} \chi(n)e^{2\pi ibn/N}
\]
and
\[
\tau(\chi) = \tau(1, \chi).
\]

The most important properties are the following.

**Lemma 11.1.** Let \( \chi \) be a Dirichlet character modulo \( N \) and \( b \in \mathbb{Z} \) such that \( (b, N) = 1 \). Then, we have \( \tau(b, \chi) = \chi(b)\tau(\chi) \).

**Proposition 11.2.** If \( \chi \) is a primitive Dirichlet character modulo \( N \) and \( b \in \mathbb{Z} \) such that \( (b, N) > 1 \), then \( \tau(b, \chi) = 0 = \chi(b)\tau(\chi) \) and
\[
|\tau(\chi)| = \sqrt{n} \quad (\neq 0).
\]

### 12 Pseudo-eigenvalues of \( W \)-operators

In this last section, we will discuss about a result due to Atkin and Li in [AL78].

Let \( N \geq 1 \) an integer. If \( N = QM \) is a product of two relatively prime integers \( Q \) and \( M \), then any Dirichlet character \( \chi \) modulo \( N \) can be expressed as \( \chi = \chi_Q \chi_M \), where \( \chi_Q \) (resp. \( \chi_M \)) is a Dirichlet character modulo \( Q \) (resp. modulo \( M \)). For \( Q \parallel N \), put
\[
w_Q = \begin{pmatrix} Qx & y \\ Nz & Qw \end{pmatrix}
\]
where \( x, y, z, w \in \mathbb{Z} \), \( y \equiv 1 \mod Q \), \( x \equiv 1 \mod N/Q \) and \( \det W_Q = Q \), and for \( f \in M_k(\Gamma_1(N)) \) set
\[
W_Q f := f|_k w_Q.
\]

**Proposition 12.1.** If \( f \in M_k(N, \chi_Q \chi_N/Q) \) (resp. \( S_k(N, \chi_Q \chi_N/Q) \)), then \( f|_k W_Q \in M_k(N, \chi_Q \chi_N/Q) \) (resp. \( S_k(N, \chi_Q \chi_N/Q) \)), and
\[
W_Q(W_Q f) = \chi_Q(-1)^{\chi_N/Q}(Q)f.
\]
In particular, the operator \( W_Q \) is independent of the choice of \( x, y, z \) and \( w \).

The following proposition, as well as the previous one, can be easily checked by straightforward computations.
Proposition 12.2. Let $p$ be a prime, $p \nmid N$ and $f \in M_k(N, \chi_{Q\chi_{N/Q}})$. Then

$$W_Q(T_p f) = \chi_Q(p) T_p(W_Q f).$$

By the latter proposition, if $f \in M_k(N, \chi_{Q\chi_{N/Q}})$ is an eigenform of the Hecke operator $T_p$ for some prime $p$ not dividing $N$ with eigenvalue $a_p(f)$, then $W_Q f$ is also an eigenform of $T_p$ with eigenvalue $\chi_Q(p) a_p(f)$. Furthermore, $W_Q$ preserves newforms. For a normalized eigenform $f \in S_k(N, \chi_{Q\chi_{N/Q}})$ there exists a normalized newform $g \in S_k(N, \chi_{Q\chi_{N/Q}})$ and a constant $\lambda_Q(f)$ such that

$$W_Q f = \lambda_Q(f) g.$$

We define $\lambda_Q(f)$ the pseudo-eigenvalue of $W_Q$ at $f$.

**Remark 12.3.** If $\chi = \chi_{\text{triv}}$, then $f = g$ and $\lambda_Q(f)$ is an eigenvalue of $W_Q$ in this case.

It is well-known that

$$W_N f = \lambda_N(f) \bar{f}$$

where $\bar{f}(z) = \sum_{n \geq 1} a_n(f) q^n$.

**Theorem 12.4.** Let $Q | N$ and $f \in S_k(N, \chi_{Q\chi_{N/Q}})$ a normalized newform. The pseudo-eigenvalue $\lambda_Q(f)$ of $W_Q$ is an algebraic number of absolute value 1.

In particular, if $\chi = \chi_{\text{triv}}$, then $\lambda_Q(f) = \pm 1$.

Let $q$ be a prime divisor of $N$ and $Q$ the $q$-primary component of $N$. If $f \in S_k(N, \chi)$ is a normalized eigenform with non-zero $q$-th Fourier coefficient $a_q(f)$, then one can express $\lambda_Q(f)$ in terms of $a(q)$ as follows.

**Theorem 12.5.** Let $f = \sum_{n \geq 1} a_n(f) q^n$ be a normalized newform with $a_q(f) \neq 0$, then

$$\lambda_Q(f) = Q^{k/2} - 1 \cdot \frac{\tau(\chi_Q)}{a_Q(f)}$$

where we use the convention that $\tau(\chi_Q) = -1$ when $Q = q$ and $\chi_Q$ is trivial modulo $Q$.

**References**
